Probability Distribution Models

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For each of the following events find the probability distribution.

1. A fair die is thrown twice, X = number of 6's thrown.

	X	P(x)
(N,N)	0	$\frac{5}{6}\times\frac{5}{6}= \quad \left(\frac{5}{6}\right)^2$
(N,6)	1	$2\times\frac{5}{6}\times\frac{1}{6}= \quad 2\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)$
(6,6)	2	$\frac{1}{6}\times\frac{1}{6}= \quad \left(\frac{1}{6}\right)^2$

2. A fair four-sided spinner numbered 1, 2, 3, 4 is spun 3 times. X = number of 1's achieved.

	X	P(x)
(N,N,N)	0	$\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}= \quad \left(\frac{3}{4}\right)^3$
(N,N,1)	1	$3\times\frac{3}{4}\times\frac{3}{4}\times\frac{1}{4}= \quad 3\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$
(N,1,1)	2	$3\times\frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}= \quad 3\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2$
(1,1,1)	3	$\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}= \quad \left(\frac{1}{4}\right)^3$

3. In an election 60% 0f the population voted. A random sample of 4 people were asked if they voted, X = number of people who voted.

	X	P(x)
(N,N,N,N)	0	$0.4\times 0.4\times 0.4\times 0.4 = \quad (0.4)^4$
(N,N,N,V)	1	$4\times(0.4\times 0.4\times 0.4\times 0.6) = \quad 4(0.4)^3(0.6)$
(N,N,V, V)	2	$6\times(0.4\times 0.4\times 0.6\times 0.6) = \quad 6(0.4)^2(0.6)^2$
(N,V, V, V)	3	$4\times(0.4\times 0.6\times 0.6\times 0.6) = \quad 4(0.4)\ (0.6)^3$
(V ,V, V, V)	4	$0.6\times 0.6\times 0.6\times 0.6 = \quad (0.6)^4$

4. A biased coin falls on heads with a probability of p and tails with a probability q. The coin is flipped 5 times and X = number of heads.

	X	P(x)
(T, T, T ,T, T)	0	$(q)^5$
(T, T, T, T,H)	1	$5(q)^4(p)$
(T, T, T,H,H)	2	$10(q)^3(p)^2$
(T, T,H,H,H)	3	$10(q)^2(p)^3$
(T,H,H,H,H)	4	$5(q)(p)^4$
(H,H,H,H,H)	5	$(p)^5$

Binomial Distribution

Each of the previous experiments is an example of a Binomial distribution.

<u>Characteristics of a Binomial Distribution</u>

A binomial distribution consists of a fixed number of trials (n) where:

1. Each trial has only two outcomes, SUCCESS (p) and FAILURE (q).
2. The probability of failure, $q = 1 - p$.
3. The outcome of successive trials is independent.
4. X is the random variable such that X is the number of successes.

When these conditions are satisfied and the random variable X follows a binomial distribution with (n) successive trials and probability of success (p) we write:

$$X \sim \text{Bin}\,(n, p)$$

The probability of X being equal to a given value (r) is defined by the formula:

$$P(X = r) = {}^{n}C_{r} \times q^{n-r} \times p^{r}$$

Examples

1. A fair die is thrown 4 times. If X represents the number of times a six occurs, find the probability of obtaining:
   (a) exactly 2 sixes (b) more than 1 six.

$$X \sim \text{Bin}\,(n=4,\ p= {}^{1}/_{6}) \qquad \text{Note: } q = {}^{5}/_{6}$$

a) $P(X = 2) = {}^{n}C_{r} \times q^{n-r} \times p^{r}$

$$= {}^{4}C_{2} \times \left(\frac{5}{6}\right)^{4-2} \times \left(\frac{1}{6}\right)^{2}$$

$$= {}^{4}C_{2} \times \left(\frac{5}{6}\right)^{2} \times \left(\frac{1}{6}\right)^{2} = \frac{25}{216} = 0.1157 \text{ (4 dp)}$$

b) $P(X > 1) = P(X = 2) + P(X = 3) + P(X = 4)$ or

$$= 1 - \left[P(x = 0) + P(X = 1)\right]$$

$$= \frac{19}{144} \quad = 0.1319 \text{ (4 dp)}$$

2. A fair coin is thrown 10 times. If X represents the number of heads, find the probability of obtaining
(a) less than 2 heads (b) at least 2 heads.

X = number of Heads

$X \sim \text{Bin}(n=10, p=0.5)$ [q = 0.5]

a) $P(X < 2) = P(X = 0) + P(X = 1)$

$$= \left({}^{10}C_0 \times 0.5^{10-0} \times 0.5^0\right) + \left({}^{10}C_1 \times 0.5^{10-1} \times 0.5^1\right)$$

$$= \left({}^{10}C_0 \times 0.5^{10}\right) + \left({}^{10}C_1 \times 0.5^9 \times 0.5^1\right)$$

$$= \frac{11}{1024}$$

$$= 0.0107$$

(b) at least 2 heads

P(X is 2 or more) = $P(X \geq 2)$

$$= 1 - P(X < 2)$$

$$= 1 - \frac{11}{1024}$$

$$= \frac{1013}{1024}$$

Example 3

A die is thrown 8 times. Calculate the probability that a score of 6 is obtained

a) exactly 3 times,
b) at least 7 times,
c) at least twice.

X = Number of 6's

$X \sim B\left(n = 8, p = \frac{1}{6}\right)$

a) P(X = 3) $= {}^{n}C_{r} \times q^{n-r} \times p^{r}$

$$= {}^{8}C_{3} \times \left(\frac{5}{6}\right)^{5} \times \left(\frac{1}{6}\right)^{3}$$

= 0.1042

b) $P(X \geq 7)$ $= P(X = 7) + P(X = 8)$

$$= \left[{}^{8}C_{7} \times \left(\frac{5}{6}\right)^{1} \times \left(\frac{1}{6}\right)^{7} \right] + \left[{}^{8}C_{8} \times \left(\frac{5}{6}\right)^{0} \times \left(\frac{1}{6}\right)^{8} \right]$$

$= 0.0000238 + 0.000000595$

$= 0.00002441$

c) $P(X \geq 2)$ $= 1 - [P(X = 0) + P(X = 1)]$

$$= 1 - \left(\left[{}^{8}C_{0} \times \left(\frac{5}{6}\right)^{8} \times \left(\frac{1}{6}\right)^{0} \right] + \left[{}^{8}C_{1} \times \left(\frac{5}{6}\right)^{7} \times \left(\frac{1}{6}\right)^{1} \right] \right)$$

$= 1-(0.232568 + 0.37210886)$

$= 0.3953$