Series

1. The Sigma Notation

   When writing a series, the shorthand Σ notation is used to represent the sum of a number of terms having a common form.

For the function $f(x)$ the series

$$f(1)+f(2)+f(3)+....+f(n-1)+f(n)$$

would be written as

$$\sum_{r=1}^{n} f(r)$$

Or simply $\sum f(r)$

Example

What expression is represented by the symbol $\sum_{r=1}^{4} 2r^2$?

$$\sum_{r=1}^{4} 2r^2 = 2 + 8 + 18 + 32 = \underline{\underline{60}}$$

2. Proof by induction

Example

Prove that $$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

For *n = 1*

$$\sum_{r=1}^{n} r^2 = 1^2 = 1 \qquad \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}1(2)(3) = 1$$

$\therefore$ Formula true for *n = 1*

Assume true for *n = k*

$$\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$$

For *n = k + 1*

$$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$$

$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

$$= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)]$$

$$= \frac{1}{6}(k+1)[2k^2 + k + 6k + 6]$$

$$= \frac{1}{6}(k+1)[2k^2 + 7k + 6]$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

$$= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$$

So if the formula works for *n = k + 1* when assumed true for *n = k*. Whatever k is.
We know it to be true for *n = 1*, hence it must be true for *n = 2*. Since true for *n = 2*, it must be true for *n = 3* and so on...

Example

Use mathematical induction to prove that, for all positive integers n,

$$2\times 2+3\times 2^2+4\times 2^3+.....+(n+1)\times 2^n=n\times 2^{(n+1)}$$

When n = 1

$$2\times 2=4 \qquad\qquad n\times 2^{(n+1)}=1\times 2^{(1+1)}=4$$

$\therefore$ Formula true for *n = 1*

Assume true for *n = k*

$$2\times 2+3\times 2^2+4\times 2^3+.....+(k+1)\times 2^k=k\times 2^{(k+1)}$$

For *n = k + 1*

$$2\times 2+3\times 2^2+4\times 2^3+.....+(k+1)\times 2^k+(k+2)\times 2^{(k+1)} = k\times 2^{(k+1)}+(k+2)\times 2^{(k+1)}$$

$$= 2^{(k+1)}\left[k+(k+2)\right]$$

$$= 2^{(k+1)}\left[2k+2\right]$$

$$= 2^{(k+2)}\left[k+1\right]$$

So if the formula works for *n = k + 1* when assumed true for *n = k*. Whatever k is.
We know it to be true for *n = 1,* hence it must be true for *all values of n >1*

Example

Prove by induction that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

When n = 1

$$\sum_{r=1}^{1} \frac{1}{r(r+1)(r+2)} = \frac{1}{1(2)(3)} = \frac{1}{6}$$

$$\frac{1}{4} - \frac{1}{2(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(2)(3)}$$

$$= \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$$

$\therefore$ Formula true for *n = 1*

Assume true for *n = k*

$$\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$$

For *n = k + 1*

$$\sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)}=\sum_{r=1}^{k}\frac{1}{r(r+1)(r+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{1}{4}-\left\{\frac{(k+3)-2}{2(k+1)(k+2)(k+3)}\right\}$$

$$=\frac{1}{4}-\left\{\frac{k+1}{2(k+1)(k+2)(k+3)}\right\}$$

$$=\frac{1}{4}-\left\{\frac{1}{2(k+2)(k+3)}\right\}$$

$$=\frac{1}{4}-\frac{1}{2((k+1)+1)((k+1)+2)}$$

So if the formula works for *n = k + 1* when assumed true for *n = k*. Whatever k is.
We know it to be true for *n = 1*, hence it must be true for *all values of n >1*