Example

Express as a single fraction $\frac{3}{(x+2)}+\frac{2}{(x-1)}$

Here the common denominator is $(x+2)(x-1)$

$$\frac{3}{(x+2)}+\frac{2}{(x-1)}=\frac{3(x-1)+2(x+2)}{(x+2)(x-1)}$$
$$=\frac{3x-3+2x+4}{(x+2)(x-1)}$$
$$=\frac{5x+1}{(x+2)(x-1)}$$

Denominator of 1st fraction is multiplied by $(x-1)$, hence multiply numerator by $(x-1)$

Denominator of 2nd fraction is multiplied by $(x+2)$, hence multiply numerator by $(x+2)$

Remove brackets on numerator and collect like terms.

Hence if we can take two fractions, or more, and write them as a single term. Then we must be able to perform the operation in reverse. This is known as placing a function as a sum of PARTIAL FRACTIONS

Partial Fractions

Rule: for each unrepeated linear factor (*ax* + *b*) in the denominator there will correspond a partial fraction $\frac{P}{(ax+b)}$

Example

Express in partial fractions $\frac{3x}{(2x-1)(x+1)}$

$$\frac{3x}{(2x-1)(x+1)}=\frac{A}{(2x-1)}+\frac{B}{(x+1)}$$

For some constant A and B.

By clearly the denominator we can use our identities, as the left hand side (LHS) must equate to the right hand side (RHS)

$$\frac{3x}{(2x-1)(x+1)} = \frac{A}{(2x-1)} + \frac{B}{(x+1)}$$
$$3x = A(x+1) + B(2x-1)$$

Look at the x. $3 = A + 2B$

Look at the constant $0 = A - B$

$\therefore A = B$

Hence solving using simultaneous equations

$$3 = A + 2B$$
$$3 = 3A$$
$$A = 1 = B$$

$$\frac{3x}{(2x-1)(x+1)} = \frac{1}{(2x-1)} + \frac{1}{(x+1)}$$

Example

Express $f(x) = \frac{1+2x}{(2+x)(1-x)}$ in partial fractions

Hence (i) Evaluate $\int_2^3 \frac{1+2x}{(2+x)(1-x)} dx$

(ii) Express $f(x)$ as a power series in x as far as the term in x^3 and give the range of convergence in the series.

$$\frac{1+2x}{(2+x)(1-x)} = \frac{A}{(2+x)} + \frac{B}{(1-x)}$$
$$1 + 2x = A(1-x) + B(2+x)$$

Look at the x. $2 = -A + B$

Look at the constant $1 = A + 2B$

Hence solving using simultaneous equations

$$3 = 3B$$
$$B = 1$$
$$A = 1 - 2 = -1$$

$$\frac{1+2x}{(2+x)(1-x)} = -\frac{1}{(2+x)} + \frac{1}{(1-x)}$$

Using partial fractions can make a function easier to integrate!

(i)

$$\int_2^3 \frac{1+2x}{(2+x)(1-x)}dx = \int_2^3 -\frac{1}{(2+x)} + \frac{1}{(1-x)}dx$$
$$= \left[- -\ln(2+x) - -\ln(1-x)\right]_2^3$$
$$= (-\ln(5) - \ln(-2)) - (-\ln(4) - \ln(-1))$$
$$= \ln 4 - \ln 5 - \ln 2$$
$$= \ln\left(\frac{2}{5}\right)$$

(ii)

$$-\frac{1}{(2+x)} = -\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$
$$= -\frac{1}{2}\left\{1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3\right\}$$
$$= -\frac{1}{2}\left\{1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8}\right\}$$
$$= -\frac{1}{2} + \frac{x}{4} - \frac{x^2}{8} + \frac{x^3}{16}$$

$$\frac{1}{(1-x)} = (1-x)^{-1}$$
$$= 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3$$
$$= 1 + x + x^2 + x^3$$

$$\therefore \ \frac{1+2x}{(2+x)(1-x)} = -\frac{1}{(2+x)} + \frac{1}{(1-x)}$$

$$= \frac{1}{2} + \frac{5}{4}x + \frac{7}{8}x^2 + \frac{17}{16}x^3$$

The expression is valid for $|x| < 1$