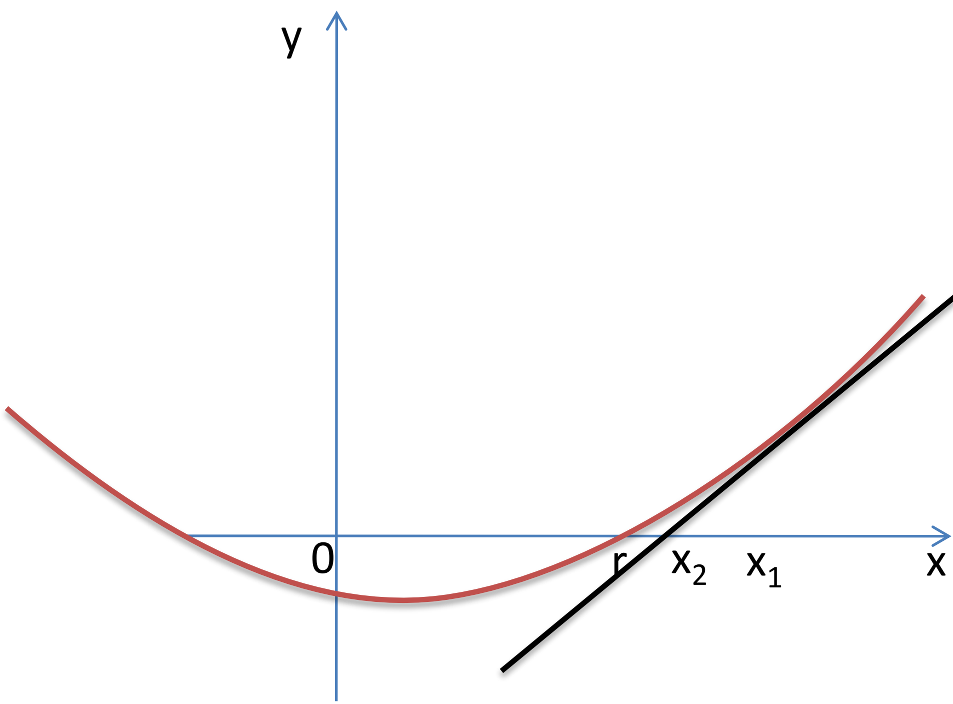
Numerical Methods

Newton Raphson method



Let $x = r$ be the root to this function $f(x)$

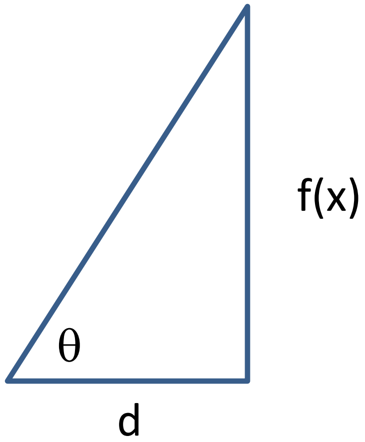
Let x_1 be a number close to r

The tangent line to the graph of $f(x)$ at $(x_1, f(x_1))$ has x_2 as its x-intercept

A better solution would therefore be at x_2.

$$x_2 = x_1 - d$$

$$d = \frac{f(x)}{\tan\theta}$$



$$\tan\theta = m = f'(x)$$

$$\boxed{x_2 = x_1 - \frac{f(x)}{f'(x)}}$$

Example

Show that the equation $e^x = 2 - x$ has only one real root and find its value using the Newton Raphson method correct to three decimal places.

$$e^x + x - 2 = 0$$

$$\therefore f(x) = e^x + x - 2$$

$$\therefore f'(x) = e^x + 1$$

$$x_0 = 1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= x_0 - \frac{\left(e^{x_0} + x_0 - 2\right)}{\left(e^{x_0} + 1\right)} = 0.5378828...$$

$$x_2 = 0.445616748$$

$$x_3 = 0.44285672$$

$$x_4 = 0.44285440$$

$$f(0.4435) = e^{0.4435} + 0.4435 - 2 = 1.65 \times 10^{-3}$$

$$f(0.4425) = e^{0.4425} + 0.4425 - 2 = -9.0 \times 10^{-4}$$

Change in sign $\Rightarrow x$ = 0.443 to 3 d.p.

<u>Example</u>

Show that the equation $2\sin x = x$ has a root between x = 1 and x = 2. Find the root correct to three significant figures.

$$2\sin x - x = 0$$

$$\therefore\ f(x) = 2\sin x - x$$

$$f(1) = 2\sin(1) - 1 = 0.68$$

$$f(2) = 2\sin(2) - 2 = 0. - 0.18$$

Change in sign $\Rightarrow$ Solution lies between x = 1 and x = 2

$$\therefore\ f(x) = 2\sin x - x$$

$$\therefore\ f'(x) = 2\cos x - 1$$

$$x_0 = 1.5$$

$$x_1 = x_0 - \frac{(2\sin x_0 - x_0)}{(2\cos x_0 - 1)} = 2.07655...$$

$$x_2 = 1.9105066$$

$$x_3 = 1.895622003$$

$$x_4 = 1.895494276$$

$$f(1.905) = 2\sin(1.905) - 1.905 = -0.0156$$

$$f(1.895) = 2\sin(1.895) - 1.895 = 8.09 \times 10^{-4}$$

Change in sign $\Rightarrow$ x = 1.90 to 3 s.f.

Example

Using Newton's method find correct to four decimal places $\sqrt[3]{3}$

Let $x = \sqrt[3]{3}$

$$x^3 = 3$$

$$x^3 - 3 = 0$$

$$\therefore f(x) = x^3 - 3$$

$$\therefore f'(x) = 3x^2$$

$$x_0 = 1.5$$

$$x_1 = x_0 - \frac{\left(x_0{}^3 - 3\right)}{\left(3x_0{}^2\right)} = 1.444444\ldots$$

$$x_2 = 1.4422529$$

$$x_3 = 1.44224957$$

$$x_4 = 1.44224957$$

$\Rightarrow x$ = 1.4422 to 4d.p.

Example

Sketch the curve with equation $y = e^x$ and on the same axes draw an appropriate line to show that the equation $e^x + x - 3 = 0$ has exactly one root α.

a) Prove that α lies between 0.7 and 0.8.

b) Taking 0.8 as a first approximation to α, use the Newton-Raphson method once to obtain a second approximation to α, giving your answer to three decimal places.

c) Show that the equation $e^x + x - 3 = 0$ can be arranged in the form $x = \ln(f(x))$

Use the iteration of the form $x_{n+1} = g(x_n)$ based on this rearrangement with $x_1 = 0.8$ to find the values of x_2 and x_3, giving your answers to three decimal places.

e) Using differentiation show that this iterative formula is convergent.