Osborn's rule

From previous examples we can see that the close comparison between identities in trigonometric functions and hyperbolic functions can be converted into a formulae known as Osborn's rule, which states that the cos should be converted to cosh and sin converted to sinh, except when there is a product of two sines, we must change the sign.

$$\cosh^2 x - \sinh^2 x = 1$$

However, whenever using Osborn's rule care must be taken as the product of two sines is sometimes disguised eg $\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$

Calculus of Hyperbolic Functions

If $f(x) = \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$

then $f'(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right) = \sinh x$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

Similarly

If $$f(x) = \sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

then $$f'(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right) = \cosh x$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

From this it follows that $$\int \sinh x\,dx = \cosh x + c$$

and $$\int \cosh x\,dx = \sinh x + c$$

Example

Differentiate

(a) $\tanh x$

(b) $\operatorname{cosech} x$

Example

Find the derivative of $\cosh 3x$ and evaluate $\int_0^{0.5} \sinh 3x \, dx$

Example

Find (a) $\int \tanh x dx$

(b) $\int \sinh^3 x dx$

Example

Integrate with respect to x $e^x \cosh x$

Inverse Hyperbolic functions

The inverse hyperbolic functions are defined in a similar manner to the inverse of trigonometric function.

If $y = \sinh x$ then $x = \sinh^{-1} y$

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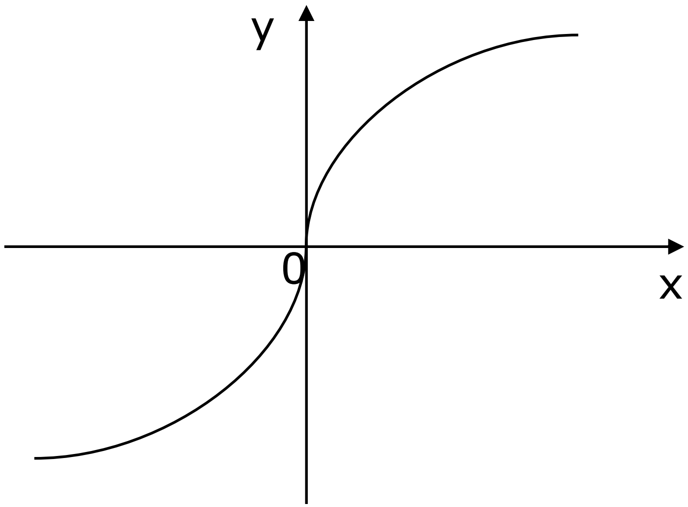
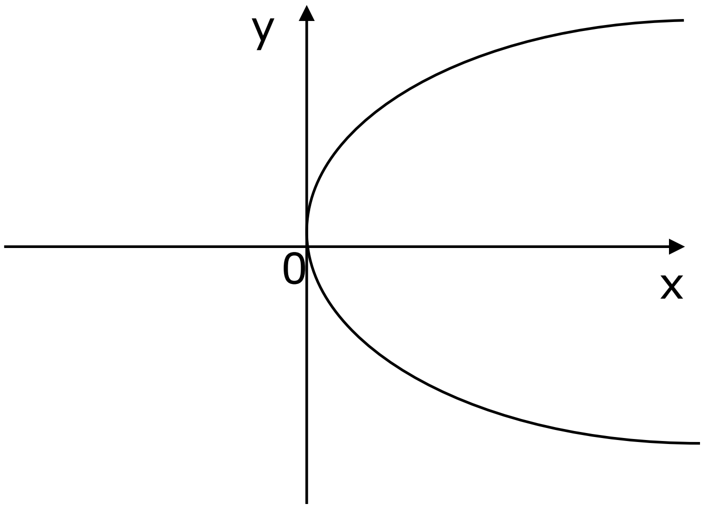
The graphs of inverse hyperbolic functions are obtained from those of the hyperbolic functions by interchanging the x and y axes.

$y = \sinh^{-1} x$



This function is a one to one function

$y = \cosh^{-1} x$



This function is a one to many function

<u>Example</u>

Let $y = \tanh^{-1} x$ so $x = \tanh y$

a) Express $\tanh y$ in the terms of e^y and hence show that

$$e^{2y} = \frac{1+x}{1-x}$$

b) Deduce that $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Derivatives of inverse Hyperbolic functions

Example

Find the derivative of $\sinh^{-1}\left(\dfrac{x}{a}\right)$ with respect of x.

Example

Find the derivative of $\cosh^{-1}\left(\dfrac{x}{a}\right)$ with respect of x.

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2 - 1}}$$

Example

Differentiate

(i) $\cosh^{-1}(2x+1)$

(ii) $\sinh^{-1}\left(\frac{1}{x}\right)$

y	$\dfrac{dy}{dx}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$

Use of Hyperbolic functions in integration

Example

Using the previous results write down the values of

(i) $$\int \frac{1}{\sqrt{x^2+1}}dx$$

(ii) $$\int \frac{1}{\sqrt{x^2-1}}dx$$

Example

a) Differentiate $\sinh^{-1}\left(\frac{x}{3}\right)$ with respect to x

b) Hence find $\int \frac{1}{\sqrt{x^2+9}}dx$

Example

Use the substitution of $x = 2\cosh u$ to show that

$$\int \frac{1}{\sqrt{x^2 - 4}}dx = \cosh^{-1}\left(\frac{x}{2}\right) + c$$

In general

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)+c$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)+c$$

Example Evaluate $\int \frac{dx}{\sqrt{3x^2-1}}$

Example

a) Express $4x^2 - 8x - 5$ in the form $A(x - B)^2 + C$ where A, B and C are constants.

b) Evaluate in terms of natural logarithms $\int_4^7 \frac{1}{\sqrt{4x^2 - 8x - 5}} dx$

Example

Evaluate $\int_{-3}^{1} \sqrt{x^2+6x+13}dx$

Leaving your answer in terms of natural logarithms.