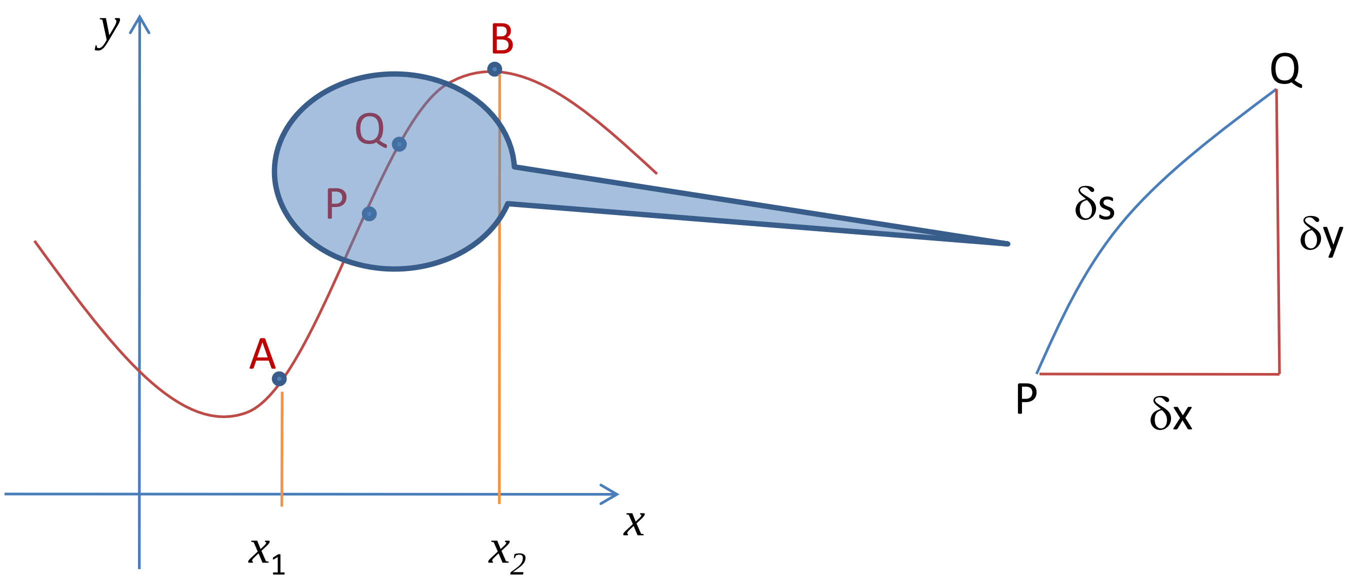
Arc length of a curve

Suppose that the arc PQ, of length δs, is such an element, then the length s, of the curve AB is given by



$$\delta x^2 + \delta y^2 = \delta s^2$$

$$1 + \frac{\delta y^2}{\delta x^2} = \frac{\delta s^2}{\delta x^2}$$

$$1 + \left(\frac{\delta y}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta x}\right)^2$$

$$\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} = \frac{\delta s}{\delta x}$$

$$\lim_{\delta x \to 0}\left(\frac{\delta s}{\delta x}\right) = \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\therefore \qquad S = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example

Find the length of the arc of the curve with equation $y = \frac{4}{3}x^{\frac{3}{2}}$ from the point where $x = \frac{3}{4}$ to the point where $x = 2$

$$y = \frac{4}{3}x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = 2x^{\frac{1}{2}}$$

$$\therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + 4x$$

$$\therefore \text{ Arc Length } = \int_{\frac{3}{4}}^{2} \sqrt{1+4x}\,dx$$

$$= \int_{\frac{3}{4}}^{2} (1+4x)^{\frac{1}{2}}\,dx$$

$$= \left[\frac{(1+4x)^{\frac{3}{2}}}{\frac{3}{2}} \times \frac{1}{4}\right]_{\frac{3}{4}}^{2}$$

$$= \left[\frac{1}{6}(1+4x)^{\frac{3}{2}}\right]_{\frac{3}{4}}^{2} = \left(\frac{9}{2}\right) - \left(\frac{4}{3}\right) = \underline{\frac{19}{6}}$$

Example

Show that in $y = a\cosh\frac{x}{a}$, the length of arc from the vertex where $x = 0$, to any point is given by $s = a\sinh\frac{x}{a}$

$$y = a\cosh\frac{x}{a}$$

$$\frac{dy}{dx} = \sinh\frac{x}{a}$$

$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\sinh^2\frac{x}{a}$$

$$= \cosh^2\frac{x}{a}$$

$$\therefore \text{ Arc Length } = \int_0^x \sqrt{\cosh^2\frac{x}{a}}\,dx$$

$$= \int_0^x \cosh\frac{x}{a}\,dx$$

$$= \left[a\sinh\frac{x}{a}\right]_0^x = a\sinh\frac{x}{a}$$

Example

Find the length of the portion of the curve $y = x^2$ between $x = 0$ and $x = 1$

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$\therefore \; 1 + \left(\frac{dy}{dx}\right)^2 = 1 + 4x^2$$

$\therefore$ Arc Length $= \displaystyle\int_0^1 \sqrt{1 + 4x^2}\, dx$

Using the substitution $x = \frac{1}{2}\sinh u$

$$\frac{dx}{du} = \frac{1}{2}\cosh u$$

$x = 0 \qquad u = 0$

$x = 1 \qquad u = \sinh^{-1} 1$

$$\therefore \text{ Arc Length } = \int_0^1 \sqrt{1+4x^2}\,dx$$

$$= \int_0^{\sinh^{-1}1} \sqrt{1+\sinh^2 u} \times \frac{1}{2}\cosh u\,du$$

$$= \int_0^{\sinh^{-1}1} \sqrt{\cosh^2 u} \times \frac{1}{2}\cosh u\,du$$

$$= \int_0^{\sinh^{-1}1} \frac{1}{2}\cosh^2 u\,du$$

$$= \frac{1}{4}\int_0^{\sinh^{-1}1} (\cosh 2u + 1)\,du$$

$$= \frac{1}{4}\left[\frac{1}{2}\sinh 2u + u\right]_0^{\sinh^{-1}1}$$

$$= \frac{1}{4}\{(1.3813\ldots) - (0)\}$$

$$= \underline{0.3453}$$