

The Standard Normal Distribution

Z – values

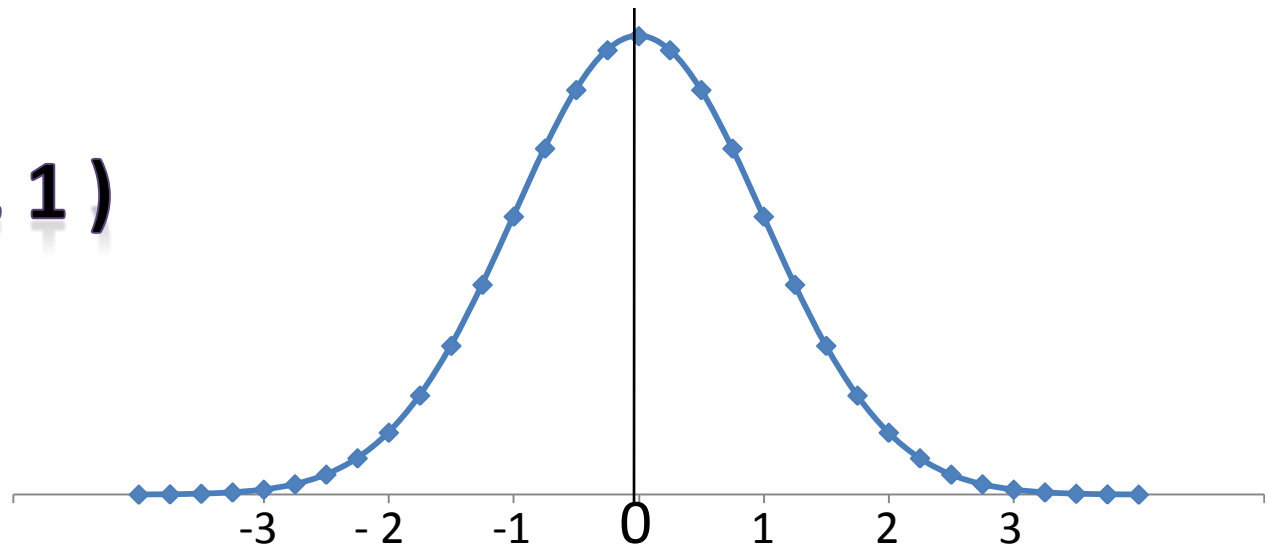
There are an infinite number of combinations of mean and variance for Normal Distribution functions but there is one particular that is very important.

This is called the Standard Normal Distribution which has a mean of zero and a standard deviation of one.

Definition:

The Standard Normal Distribution is a normal distribution with a random variable z , and a mean of 0 and standard deviation of 1.

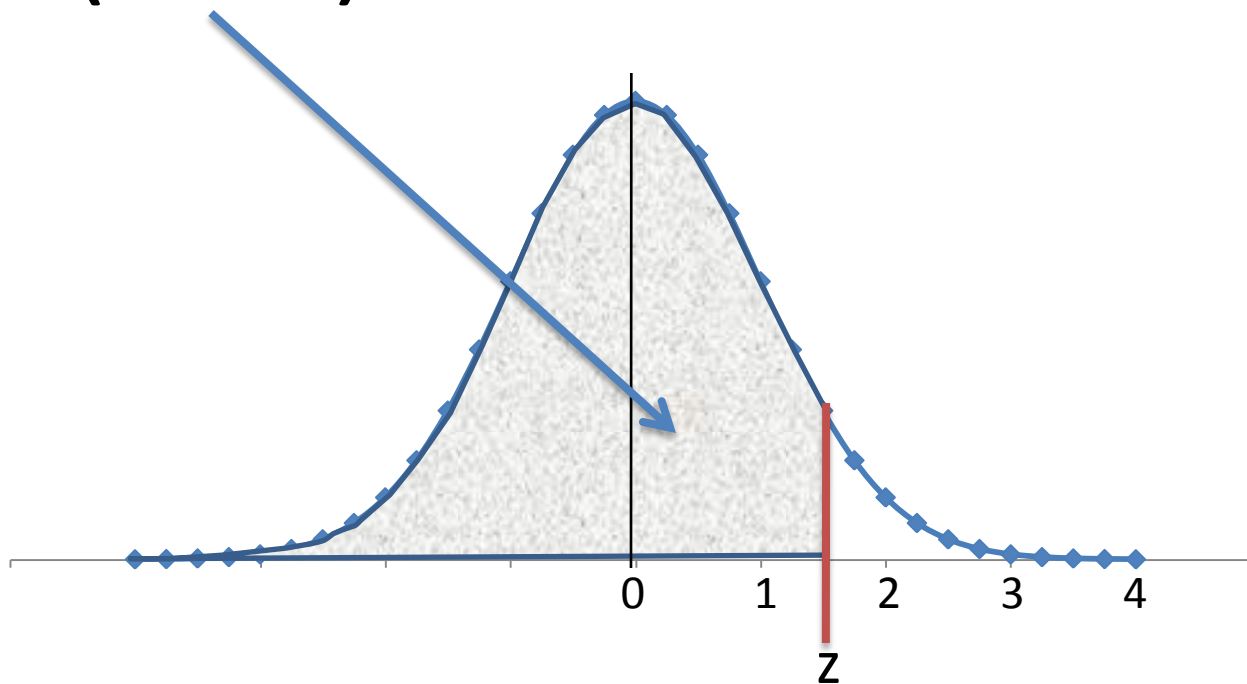
$$Z \sim N(0, 1)$$



For this Standard Normal Distribution the possible probabilities are calculated and presented in what is commonly known as Z – tables, which are included in your statistical tables booklet.

The table only contains probabilities such that:

$$P(X \leq z) \quad \text{For values of } z \text{ from } 0 \text{ to } 4$$



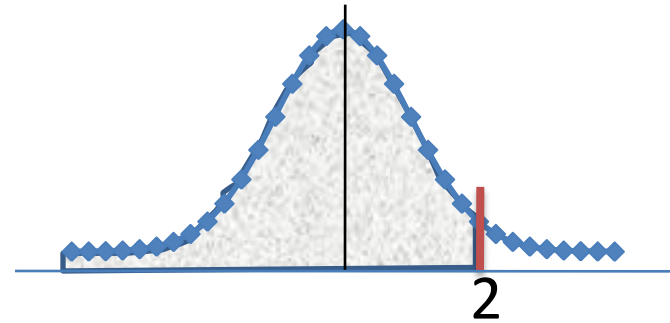
However, using the symmetrical nature of the graph it is possible to calculate probabilities if z is negative and other variations.

Example:

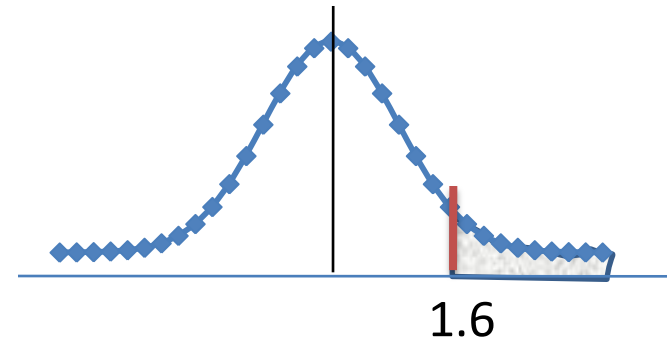
From the Standard Normal Distribution tables (Z – Tables), find

- a) $P(z < 2)$ (b) $P(z > 1.6)$ (c) $P(z < -2)$ (d) $P(1 < z < 2.3)$

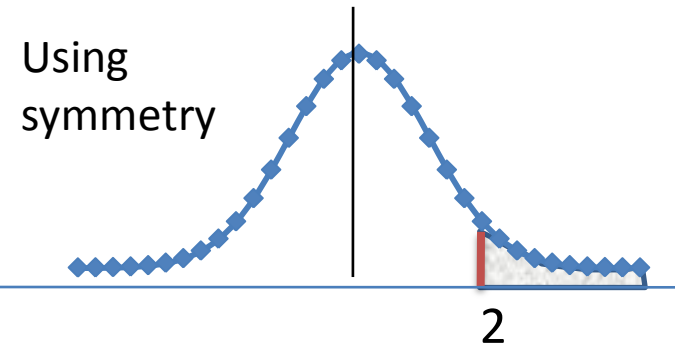
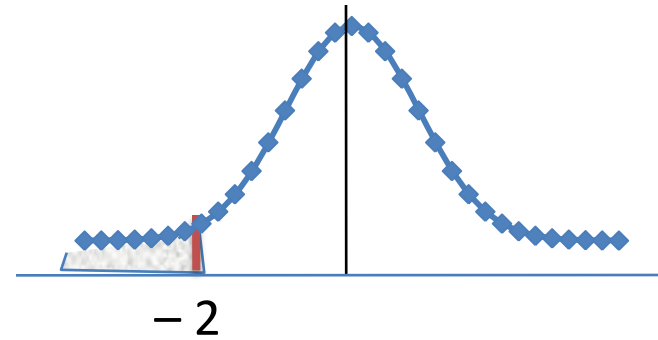
a) $P(z < 2) = 0.97725$



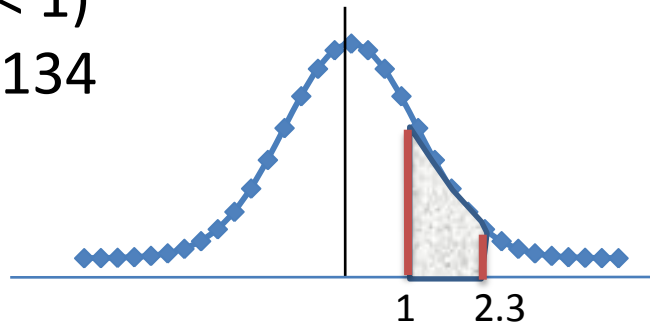
b) $P(z > 1.6) = 1 - P(z < 1.6)$
 $= 1 - 0.9452$
 $= 0.0548$



$$\begin{aligned}
 \text{c) } P(z < -2) &= P(z > 2) \\
 &= 1 - P(z < 2) \\
 &= 1 - 0.97725 \\
 &= 0.02275
 \end{aligned}$$



$$\begin{aligned}
 \text{d) } P(1 < z < 2.3) &= P(z < 2.3) - P(z < 1) \\
 &= 0.98928 - 0.84134 \\
 &= 0.14794
 \end{aligned}$$



Finding a z – value from a given probability

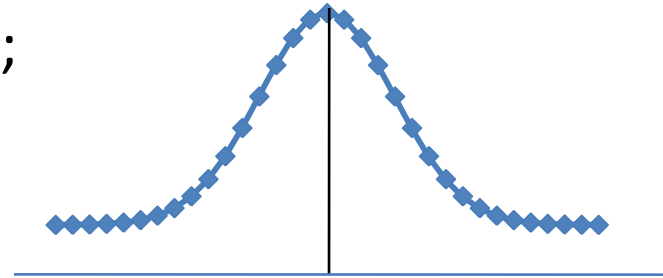
We can be asked to determine a z – value from a given probability value.

Example:

If $Z \sim N(0, 1)$ find the value of c when;

(a) $P(Z < c) = 0.75$

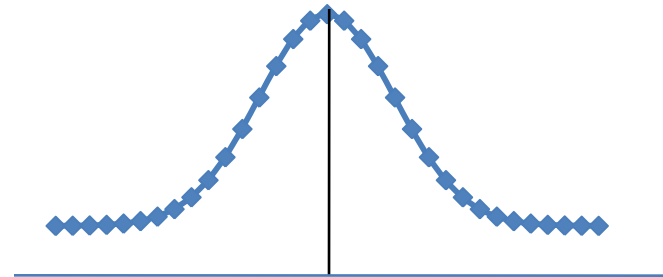
$$c = 0.674$$



(b) $P(Z > c) = 0.31$

$$\Rightarrow P(Z < c) = 0.69$$

$$c = 0.496$$

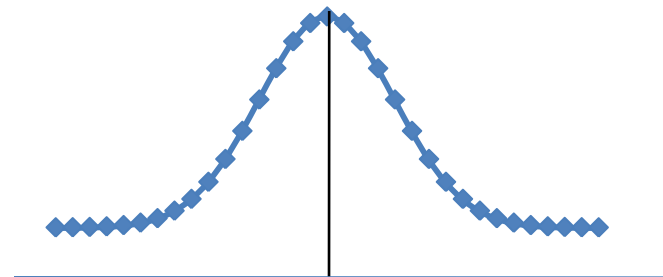


(c) $P(Z < c) = 0.12$

$$\Rightarrow P(Z > -c) = 0.12$$

$$\Rightarrow P(Z < -c) = 0.88$$

$$c = -1.175$$



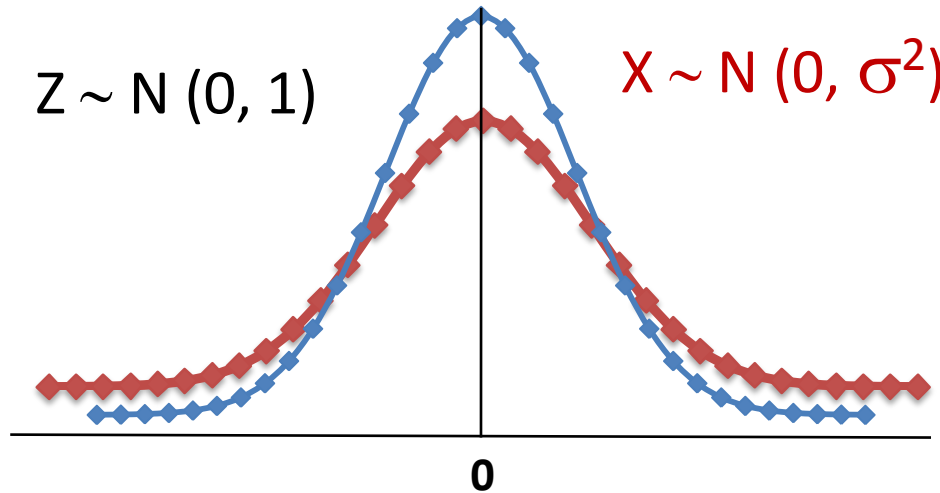
As mentioned previously there is an infinite combination of possible means and variances for normally distributed variables and it is impossible to create tables of probability for each combination.

However, it is possible to change a given Normal distribution, using a simple transformation.

Changing to a Standard Normal Distribution

To change the Standard Normal Distribution $Z \sim N(0, 1)$ into a general distribution $X \sim N(\mu, \sigma^2)$ we must:

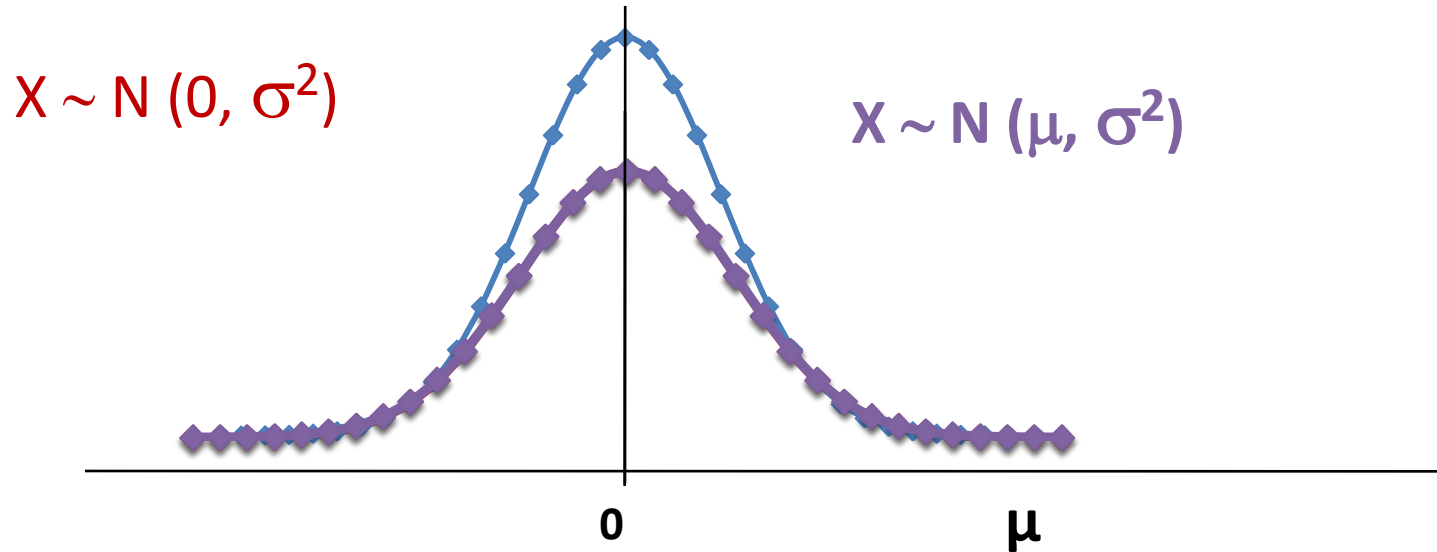
1. Multiply Z by σ . $X = Z\sigma$ This transformation increases the variance by a scale factor of σ^2 .
New variance = $1 \times \sigma^2 = \sigma^2$



2. Add μ to $Z\sigma$

$$X = Z\sigma + \mu$$

This transformation shifts the mean variance from 0 to μ .
New mean = $0 + \mu = \mu$



The transformation $X = Z\sigma + \mu$ will now change $Z \sim N(0, 1)$ into any normal distribution $X \sim N(\mu, \sigma^2)$.

We can now make Z the subject of this formula,

$$X = Z\sigma + \mu$$

$$X - \mu = Z\sigma$$

$$\frac{X - \mu}{\sigma} = Z$$

So in reverse to change any normal distribution to a Standard Normal Distribution is:

$$z = \frac{X - \mu}{\sigma}$$

We can now use the transformation and the z – tables to calculate the probability for any normal distribution.

Example:

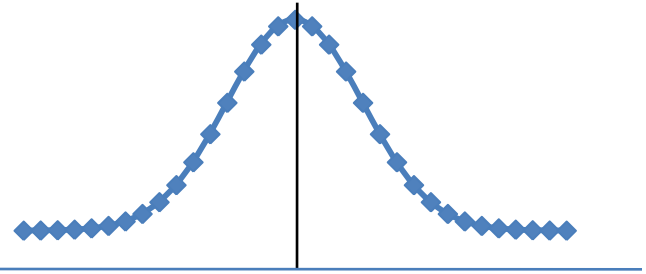
Given that $X \sim N(80, 16)$ find (a) $P(X > 90)$ (b) $P(X < 68)$ (c) $P(72 < x < 85)$

$$\begin{aligned} \text{a) } P(X > 90) &= P(z > 2.5) \\ &= 1 - P(z < 2.5) \\ &= 1 - 0.99379 \\ &= 0.00621 \end{aligned}$$

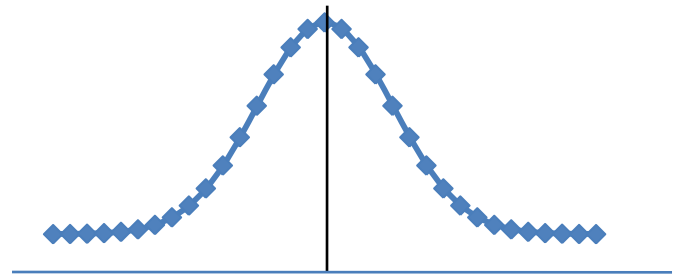
$$\begin{aligned} \text{b) } P(X < 68) &= P(z < -3) \\ &= 1 - P(z < 3) \\ &= 1 - 0.99865 \\ &= 0.00135 \end{aligned}$$

$$z = \frac{X - \mu}{\sigma} = \frac{90 - 80}{\sqrt{16}} = 2.5$$

Note: $\sigma = \sqrt{16}$ as $\sigma^2 = 16$



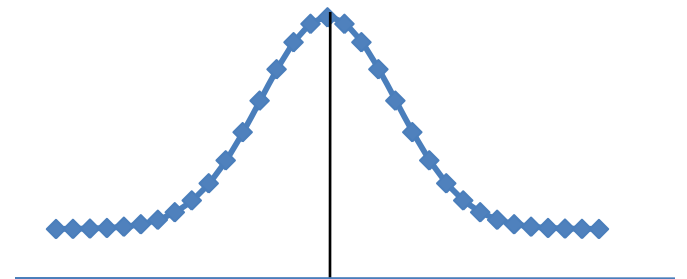
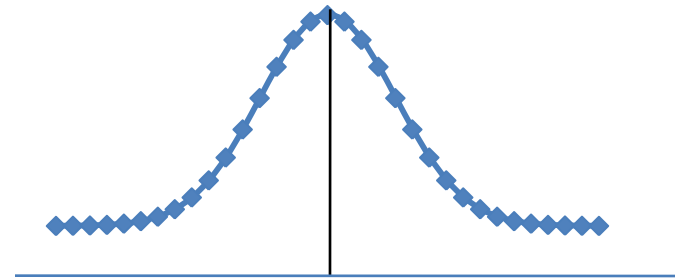
$$z = \frac{X - \mu}{\sigma} = \frac{68 - 80}{\sqrt{16}} = -3$$



$$\begin{aligned} \text{c) } P(72 < x < 85) &= P(-2 < z < 1.25) \\ &= P(z < 1.25) - P(z < -2) \\ &= P(z < 1.25) - [1 - P(z < 2)] \\ &= 0.89435 - [1 - 0.97725] \\ &= 0.89435 - 0.02275 \\ &= 0.8716 \end{aligned}$$

$$Z_1 = \frac{X - \mu}{\sigma} = \frac{85 - 80}{\sqrt{16}} = 1.25$$

$$Z_2 = \frac{X - \mu}{\sigma} = \frac{72 - 80}{\sqrt{16}} = -2$$

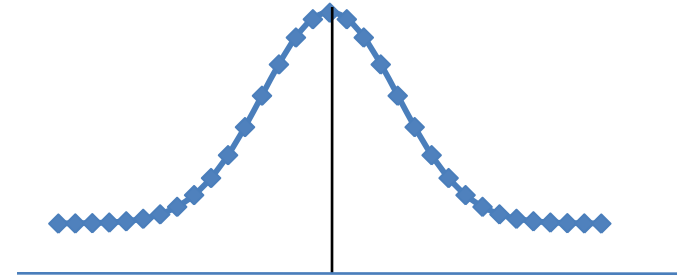


Examples.

1. Obtain $P(X < 13)$ from $X \sim N(10, 4)$

$$z = \frac{X - \mu}{\sigma} = \frac{13 - 10}{\sqrt{4}} = 1.5$$

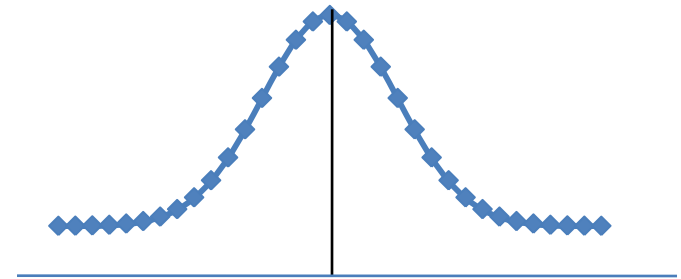
$$\begin{aligned} \text{a) } P(X < 13) &= P(z < 1.5) \\ &= 0.93319 \end{aligned}$$



2. Obtain $P(X < 11)$ from $X \sim N(15, 4)$

$$z = \frac{X - \mu}{\sigma} = \frac{11 - 15}{\sqrt{4}} = -2$$

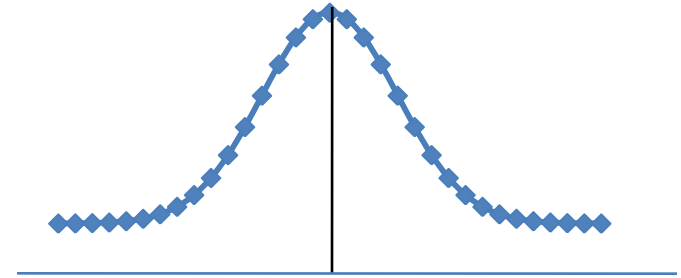
$$\begin{aligned} \text{a) } P(X < 11) &= P(z < -2) \\ &= 1 - P(z < 2) \\ &= 0.0228 \end{aligned}$$



3. Obtain $P(X > 5)$ from $X \sim N(7, 9)$

$$z = \frac{X - \mu}{\sigma} = \frac{5 - 7}{\sqrt{9}} = -0.67$$

$$\begin{aligned} \text{a) } P(X > 5) &= P(z > -0.67) \\ &= P(z < 0.67) \\ &= 0.74857 \end{aligned}$$



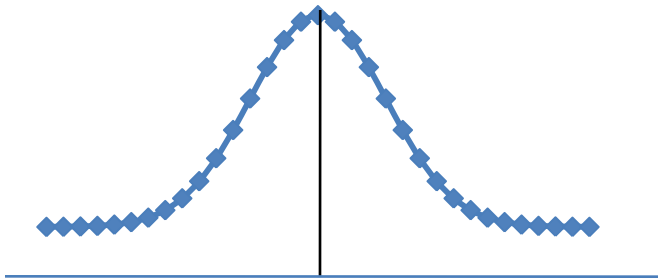
Further Examples

1. An electrical firm manufacturers light bulbs with a life span which is normally distributed with a mean of 800 hours and a variance of 1500 hours². Find the probability that a given bulb will have a life span between 780 and 840 hours.

X = Life span of a bulb

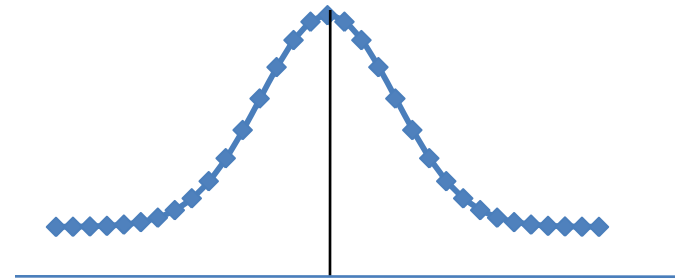
$X \sim N(800, 1500)$

$$\begin{aligned} P(780 < x < 840) &= P(-0.52 < z < 1.03) \\ &= P(z < 1.03) - P(z < -0.52) \\ &= P(z < 1.03) - [1 - P(z < 0.52)] \\ &= 0.84849 - [1 - 0.69847] \\ &= 0.84849 - 0.30153 \\ &= 0.54696 \end{aligned}$$



$$Z_1 = \frac{X - \mu}{\sigma} = \frac{780 - 800}{\sqrt{1500}} = -0.52$$

$$Z_2 = \frac{X - \mu}{\sigma} = \frac{840 - 800}{\sqrt{1500}} = 1.03$$



2. The contents of a can are designed to have a mean weight of 15.7 oz. The packing is done by machine and the weight may vary from can to can but is normally distributed with a standard deviation of 0.12 oz.

If 1000 cans are tested how many would you expect to weigh less than 15.5 oz.

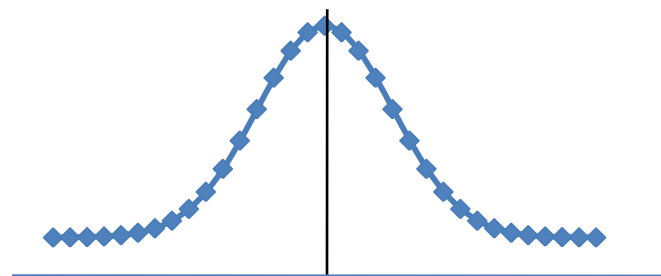
X = Weight of the contents of the can

$X \sim N(15.7, 0.12^2)$ [Note: $\sigma = 0.12$, variance (σ^2) = 0.12^2]

$$\begin{aligned} P(X < 15.5) &= P(z < -1.67) \\ &= 1 - P(z < 1.67) \\ &= 1 - 0.95254 \\ &= 0.04746 \end{aligned}$$

$$\begin{aligned} \therefore \text{Expected number} &= 1000 \times 0.04746 \\ &= 47 \text{ cans} \end{aligned}$$

$$z = \frac{X - \mu}{\sigma} = \frac{15.5 - 15.7}{0.12} = -1.67$$



3. A lawyer commutes daily to work and on average the trip takes 24 minutes with a standard deviation of 3.8 minutes. The time for all journeys are normally distributed.

- a) Find the length of time below which we find 75% of the journeys.
- b) Find the length of time above which we find the fastest 15% of the journeys.
- c) Find the probability that in any 5 day working week at least four of the journeys take no more than 1/2 an hour.

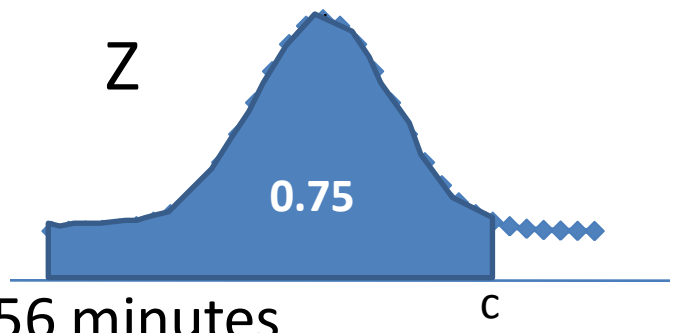
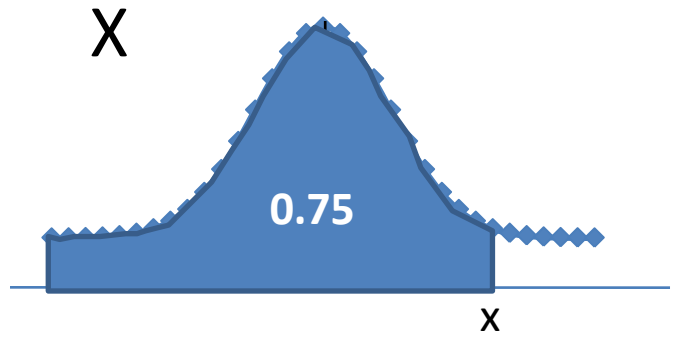
$X = \text{Length of journey}$ $X \sim N(24, 3.8^2)$

(a) $P(X < x) = 0.75$

$\Rightarrow P(Z < c) = 0.75$
 $c = 0.674$ [from tables]

Using: $Z = \frac{X - \mu}{\sigma}$
 $0.674 = \frac{x - 24}{3.8}$

$x = 24 + (3.8 \times 0.674) = 26.56$ minutes

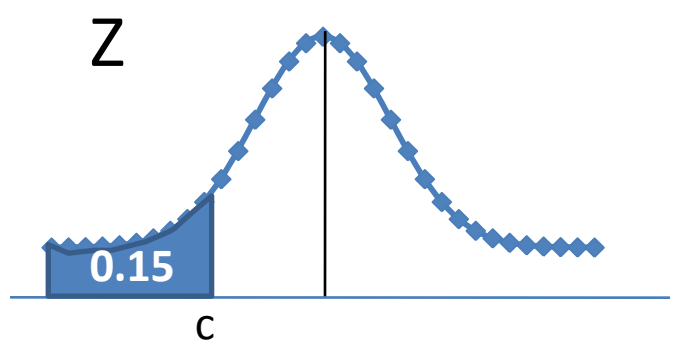
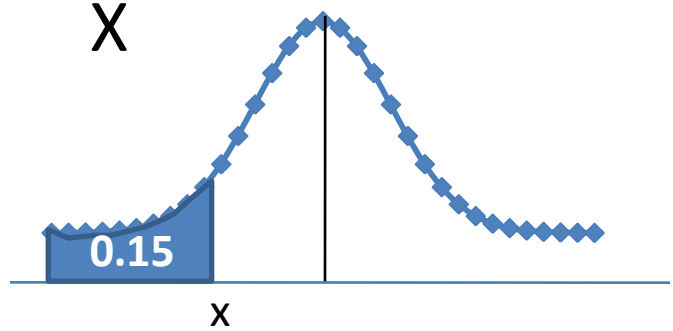


(b) $P(X < x) = 0.15$
 $\Rightarrow P(Z < c) = 0.15$
 $\Rightarrow P(Z > -c) = 0.15$
 $\Rightarrow P(Z < -c) = 0.85$
 $c = -1.036$

Using: $Z = \frac{X - \mu}{\sigma}$

$$-1.036 = \frac{x - 24}{3.8}$$

$$x = 24 + (3.8 \times -1.036) = 20.06 \text{ minutes}$$



$$\begin{aligned} \text{c) } P(X < 30) &= P(z < 1.58) \\ &= 0.9428 \end{aligned}$$

$$z = \frac{X - \mu}{\sigma} = \frac{30 - 24}{3.8} = 1.58$$

Let Y = number of journeys taking less than $\frac{1}{2}$ an hour

$$Y \sim \text{Bin} (n = 5, p = 0.9428)$$

$$\begin{aligned} P(Y \geq 4) &= P(Y = 4) + P(Y = 5) \\ &= [{}^5C_4 \cdot (0.0572)^1 \cdot (0.9428)^4] + [{}^5C_5 \cdot (0.0572)^0 \cdot (0.9428)^5] \\ &= 0.97 \text{ (2 d.p.)} \end{aligned}$$